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FRACTIONAL DERIVATIVES AND FRACTIONAL INTEGRALS (THE WORK OF SCIENCE)

Science

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ABSTRACT

We discuss regular approaches to the problems and definition of the fractional derivatives and fractional integrals (simply called differ integrals), namely the Riemann-Liouville Fractional derivative and Caputo fractional derivative and fractional integrals. We prove the basic properties of fractional integrals and Fractional derivatives as well as some theorems of the fractional integrals and derivatives including the rules for their compositions and the conditions for the equivalence of various definitions.

The paper focuses on find the approximate values for functions derivatives, when the function order is a negative, illustrate by a some theorems and examples.

KEYWORDS

Mittag-Leffler functions, Gamma functions, Beta functions and their properties are briefly discussed.

INTRODUCTION TO FRACTIONAL CALCULUS

Fractional Calculus is important branch of mathematics. The fractional calculus is more than 300 years old. Its generalization of ordinary differential and integral is non-integer (Arbitrary) order. The subject is as old as the calculus differentiation and goes to back to time when Leibniz, Gauss, and Newton invented this kind calculation in letter to L-hospital in 1695 Leibniz raise the following equation.

Millerand Ross, 1693 can the meaning of derivatives with integer order to be generalized to derivatives with non-integer order?

The story goes that L hospital was somewhat curios about the equation and replied another to Leibniz what. If the order be ½?

Leibniz in a letter dated September 30 /1695 replied. It will lead to a paradox which one day useful consequence will be drawn the equation raised for fractional derivatives was going a topic the last 300 years. Several mathematicians contributed to this subject over the year people like Lowville, Riemann and well mad major contribution to the theory of fractional calculus. The Several mathematicians contributed to this subject over the year people like Lowville, Riemann and well mad major contribution to the theory of fractional calculus. story of fractional calculus continued with contribution from Fourier, Abel, Leibniz, Grunewald and Letnikov. Now days the fractional calculus attract many scientist and engineers there are several application of this phenomenon in mathematics ,physics, chemistry, control theory and so on.

The fractional calculus is a natural extension of that traditional calculus. It is like many other mathematical branches and ideas; it has its origin in the pursing for the extension of meaning. Well known examples are the extension of the integer number to the rational number of the rational numbers to the real numbers, and of the real number numbers to the complex number. The question of extension of meaning in differential and integral calculus: can the derivative $\frac{d^n y}{d^n x}$ of integer order,n>0, be extended to any order, n, functional, irrational or complex? The answer of this question has led to the development of a new theory which is called fractional calculus.

1.1 IMPORTANCE OF FRACTIONAL CALCULUS

Until recent times, it was considered that fractional calculus is understood by only for the few selected mathematics who have spatial calculus knowledge in this and also that it was considered that is only mathematical theory without application, but in the last few decades there has been an explosion of the research activities of the application of the fractional calculus to very diverse scientific field ranging from the physical phenomena to control system, to finance and to economic. Virtually no area of classical analysis is left untouched by fractional calculus, indeed at present applications and activities related to fractional calculus have appeared in at least the following fields: in fractional control of engineering system and advancement of calculus of variation and optimal control to dynamic system, in analytical and numerical tools and technique , in fundamental exploration of mechanical electronic , and thermal constituted relation and other properties of varies engineering material such as viscoelastic polymers, …

1.2 HISTORICAL DEVELOPMENT OF FRACTIONAL CALCULUS

The concept of fraction calculus is believed to have emerged from a question raised in the year1695. Marquis de L Hopital in the latter date of September 30th, 1695 l Hopital wrote to lionize asking about him particular notation he had used in this publications for the $n-th$, derivative $\frac{d^n f(x)}{dx^n}$ of the linear function $f(x) = x$ L Hospital posed the equation, that $d^{\frac{1}{2}}x$ will be equal to, $x\sqrt{dx:x}$". in this word fractional calculus was born. Following this equation, many mathematicians contributed to the fractional calculus in 1730, Euler mentioned interpolating between integral orders of derivative .In 1812Laplace defined fractional derivative by means of an integral and in 1819 there appeared the first discussion of fractional derivative in a calculus text written by S.F.Lacroix.

Starting with

$$y = x^m \qquad (1.1)$$

Where m is positive integer, Lacroix found that $n-th$ derivative of x^m

$$\frac{D^n f}{D^n x} = \frac{m!}{(m-n)!}x^{m-n}, m \geq n. \qquad (1.2)$$

Then replaced n *with* $\frac{1}{2}$ *and let* $m = 1$ *thus the derivative of order* $\frac{1}{2}$ of the function x is

$$\frac{d^{1/2}}{dx^{1/2}}x = \frac{2\sqrt{x}}{\sqrt{\pi}} \qquad (1.3)$$

This result obtained by Lacroix is the same as that yielded by the present day Riemann- Liouville definition of the fractional derivative .But Lacroix consider the equation of interpolating between integral orders and of a derivative. He developed only two of the 700 pages of this text to this topic.

Fourier, in 1822, was the next to mention a derivative of arbitrary order. But like Euler, Laplace, and Lacroix, he gave no application. The first use of fractional operation was by Neil Henrik Abel in 1823. [21]Abel applied the fractional calculus to the solution of an integral equation, which arose in his formulation of the tautochrone problem. To find the shape of a frictionless wire lying in a vertical plane, such that the time required for a bead placed on the wire to slide to the lowest point of the wire is the same regardless of where the bead is first placed.

Probably Joseph Liouville and fascinated by Laplace's and Fourier's brief comments or Abel's solution, so he made the first major study of fractional calculus. He published three large memoirs on this topic in 1832 beginning with followed by more papers in rapid succession. Liouville first definition of a derivative of arbitrary order v involved an infinite series. This had the disadvantage that v must be restricted to those values for which the series converges. Liouville seemed aware of the restrictive nature of his first definition, therefore. Liouville tried to put his effort to define fractional derivatives again of x^{-a} whenever x and a are positive.

Starting with a definite integral we have:

$$I=\int_0^{\infty} u^{a-1}e^{-xu}du \qquad (1.4)$$

With the change of variables $xu=t$, we obtain

$$I=x^{-a}\int_0^{\infty} t^{a-1}e^{-t}dt$$

This integral closed to the Gamma integral of Euler which is define as

$$\Gamma(a)=\int_0^{\infty} t^{a-1}e^{-t}dt$$

Therefore the equation (1.4) con be written in the tern of Legendre symbol Γ for generalized factorial

$$I=x^{a}\Gamma(a)$$

Which implies

$$x^{-a}=\frac{1}{\Gamma(a)}$$

By" operating "on both side of this equation with d^v/dx^v , and by assuming that

$$\frac{d^v(e^{ax})}{dx^v}=e^{ax}a^v \quad \text{For any } v>0\text{, } Liouville \text{ was able to obtain.}$$

The result known as this section definition

$$\frac{d^v}{dx^v}x^{-a}=\frac{(-)^v\Gamma(a+v)}{\Gamma(a)}x^{-a-v}$$

(1.5)

After these attempt, still the second definition of the fractional derivative is restricted to some functions like, $f(x)=x^{-a}$. The $(-1)^v$ tern in this expression suggests the need- to extend the theory to include complex numbers. Indeed, Liouville was able to extend this definition to include complex value for a and v. By piecing together the somewhat disjointed accomplishments of many notable mathematicians, especially Liouville and Riemann, modern analysts can now define the integral of arbitrary order. The fractional integral of order v is defined as follows.

$$D^{-v}f(t)=\frac{1}{\Gamma f(v)}\int_a^x f(t)(x-t)^{v-1}dt \qquad (1.6)$$

1.3 SPECIAL FUNCTIONS

There are some basic mathematical functions which are important in the study of the theory of the Fractional Calculus. In the next subsection, we will concentrate on the Gamma function, Beta function, Mittag-leffler function…
And we, will study some well-known properties of this functions.

1.4 EULER'S GAMMA FUNCTION

One of the basic special function is Euler Gamma function .this function is tied to fractional calculus by definition we will see later on the fraction integration.

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)},,\alpha,\beta>0$$

And also, we can be written in the term one parameter of Mittag-Leffler function

$$E_{1(x)=}E_{1,1}(x)=e^x$$

PROOF: Let as using the definition of Mittag-Leffler function

1: if $\alpha=1, and\ \beta=1\ then\ the\ mittafunction\ will\ come$

$$E_{1,1}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{x^k}{(k)!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^k}{k!}=e^x$$

If we let $x=0$,than

In1729, Euler discovered function. There are several approaches leading to the definition of gamma function. The most preferred way of defining it, is the use of Euler's integral

The function $\Gamma: R^+\to R^+$ defined by the integral

$$\Gamma(x)=\int_0^{\infty} e^{-x}x^{n-1}dx \quad ,x>0$$

SOME PROPERTIES OF GAMMA FUNCTION AS FOLLOW

Its relation to the factorials is that for any natural number, n , we have $\Gamma(n)=(n-1)!$

The Gamma function satisfies the following function equation

$$\Gamma(x+1)=x\Gamma(x) \qquad x\epsilon R^+$$

The Gamma function satisfies the following function

$$\Gamma(n)=z^n\int_0^{\infty} e^{-zx}x^{n-1}dx$$

And also the following property is hold

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

In 1730, Euler generalized the formula

$$\frac{d^m}{dx^n}x^m=\frac{m!}{(m-n!}x^{m-n}$$

By using the following property

$$\Gamma(m+1)=m!$$

We obtain

$$\frac{d^m}{dx^n}x^m=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

The incomplete Gamma function is more generalized form of the Gamma function, it is given by

$$\Gamma^*(v,t)=\frac{1}{\Gamma(v)t^v}\int_v^t e^{-x}x^{v-1}dx\ ,\ \ Rev>0.$$

1.5 BETA FUNCTION

One of the useful mathematical functions in fractional calculus is the Beta function. Its solution is defined through the use of multiple Gamma functions. Also, it share form that is characteristically similar to the fractional integral or derivative of many functions particular polynomials of the form t^{α} . The Beta function is defined by definite integral. The following equation demonstrates the Beta integral and its solution in terms of the Gamma function

$$\beta(n,m)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad n,m>0 \qquad (1.7)$$

SOME PROPERTIES OF BETA FUNCTION

Symmetric functions of Beta

$\beta(n,m)=\beta(m,n)$

The beta function can also be defined in the terms of the Gamma function

let $Re(x)>0$, $and\ Re(y)>0$

Than $\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\beta(y,x)$

the beta function relation with the Beta function

$$B(x+1,y)=\frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+y+1)}=\frac{x}{x+y}B(x,y) \qquad \text{where } x,y \in R^+$$

(1.8)

1.6 THE MITTAG -LEFFLER FUNCTION

Another important function of fractional calculus is the Mittag-Lellfer function which is plays a signification role in the solution of non-integral order differential equations. The one parameter representation of Mittag-Leffler function is defined over the entire complex by plane

$$E_{\alpha}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}\ ,\quad \alpha>0, x\epsilon\mathbb{C}$$

And also two parameter representation of the mittag-leffler function

$$\frac{1}{\alpha x}\left[E_{\alpha,\beta-1}(x)-(\beta-1)E_{\alpha,\beta}(x)\right]=\frac{1}{\alpha d}\left[\sum_{k=0}^{\infty}\frac{(\alpha k)x^k+\beta x^k-x^k}{\Gamma(\alpha k+\beta-1)}-\sum_{k=0}^{\infty}\frac{\beta x^k-x^k}{\Gamma(\alpha k+\beta)}\right]$$

$$\frac{1}{\alpha x}\left[E_{\alpha,\beta-1}(x)-(\beta-1)E_{\alpha,\beta}(x)\right]=\sum_{k=0}^{\infty}\frac{(\alpha k)x^k}{\Gamma(\alpha k+\beta)}\left(\frac{1}{\alpha x}\right).$$

Hence, we conclude that

$$\frac{d}{dx}\left[E_{\alpha,\beta}(x)\right]=\frac{1}{\alpha x}\left[E_{\alpha,\beta-1}(x)-(\beta-1)E_{\alpha,\beta}(x)\right] \qquad (1.11)$$

Property 4: If $\alpha=1\ and\ \beta=2$ then the Mittag-Leffler function be come, $\frac{e^x-1}{x}$

$E_{\alpha,\beta}(0) = 1$

The Mittag-Leffler function is the extension of the exponential function, solution of the fractional order differential equations are often expressed in the term of Mittag-Leffler functions in much the same way to solutions of many integer order differential equations may be expressed in terms of exponential functions. The following some

PROPERTIES OF MITTAG -LEFFLER FUNCTION

Property 1. $\frac{1}{\Gamma(\beta)} + x\, E_{\alpha,\beta}(x) = E_{\alpha,\beta}(x)$, where $x\epsilon\mathbb{C}$

Proof Using the definition of Mittag-Leffler function, the proof can be show directly as below:

We will proceed the proof by using the definition of Mittag-Leffler function.

$$\frac{1}{\Gamma(\beta)} + xE_{\alpha,\beta}(x) = \frac{1}{\Gamma(b)} + x\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\alpha+\beta)}$$

$$\frac{1}{\Gamma(\beta)} + xE_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} + \sum_{k=0}^{\infty}\frac{x^{k+1}}{\Gamma(\alpha k+\alpha+\beta)}$$

If we let $m = k + 1$, in the above sum notation, we obtain

$$\frac{1}{\Gamma(\beta)} + xE_{\alpha,\alpha+\beta}(x) = \frac{1}{\Gamma(\beta)} + \sum_{m=0}^{\infty}\frac{x^m}{\Gamma(\alpha m+\beta)}$$

$$\frac{1}{\Gamma(\beta)} + xE_{\alpha,\alpha+\beta}(x) = \frac{1}{\Gamma(\beta)} + \sum_{m=0}^{\infty}\frac{x^m}{\Gamma(\alpha m+\beta)} - \frac{1}{\Gamma(\beta)}$$

$$\frac{1}{\Gamma(\beta)} + xE_{\alpha,\alpha+\beta}(x) = E_{\alpha,\beta}(x)$$

Property 2. $e^x[1 + E_r f(x)] - 1 = xE_{1/2,3/2}(x),$ (1.9)

Proof

by using the definition of fractional Mittag-Leffler function, we have

$$E_{1/2}(x) = e^{x^2}\left[1 + \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz\right]$$

$$E_{1/2}(x) = e^{x^2}[1 + E_r f(x)].$$

From the equation N0 (2.17) it can be seen easily that

$$xE_{1/2,3/2}(x) = xE_{1/2,1}(x) - 1 = xE_{1/2,}(x) - 1$$

By observing the above two equations, we may conclude that

$$e^x[1 + E_r f(x)] - 1 = xE_{1/2,3/2}(x),$$

Property 3

$$\frac{d}{dx}[E_{\alpha,\beta}(x)] = \frac{1}{\alpha x}[E_{\alpha,\beta-1}(x) - (\beta-1)E_{\alpha,\beta}(x)] \quad (1.10)$$

Proof

By looking at the left hand side of the equation number (1.10)

$$\frac{d}{dx}[E_{\alpha,\beta}(x)] = \frac{d}{dx}\left[\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}\right]$$

$$\frac{d}{dx}[E_{\alpha,\beta}(x)] = \sum_{k=0}^{\infty}\left[\frac{1}{\Gamma(\alpha k+\beta)}\right]\frac{d}{dx}x^k$$

$$\frac{d}{dx}[E_{\alpha,\beta}(x)] = \sum_{k=0}^{\infty}\frac{kx^{k-1}}{\Gamma(\alpha k+\beta)}$$

Then the right hand of the equation No (2.28) we, have

$$\frac{1}{\alpha x}[E_{\alpha,\beta-1}(x) - (\beta-1)E_{\alpha,\beta}(x)] = \frac{1}{\alpha d}\left[\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta-1)} - (\beta - 1)\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}\right]$$

$$\frac{1}{\alpha x}[E_{\alpha,\beta-1}(x) - (\beta-1)E_{\alpha,\beta}(x)] = \frac{1}{\alpha d}\left[\sum_{k=0}^{\infty}\frac{x^k(\alpha k+\beta)}{\Gamma(\alpha k+\beta-1)(\alpha k+\beta)} - \sum_{k=0}^{\infty}\frac{x^k(\beta-1)}{\Gamma(\alpha k+\beta)}\right]$$

PROOF: Let $f(t)$ and $g(t)$ are tow fractional integral functions

And λ is some constant than, using the definition of fractional integrals

$$D_a^{\alpha}[\lambda f(y) + \lambda g(y)] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-y)^{\alpha-1}[\lambda f(y) + \lambda g(y)]d\,y$$

$$D_a^{\alpha}[\lambda f(y) + \lambda g(y)] = \lambda\frac{1}{\Gamma(\alpha)}\int_a^t (t-y)^{\alpha-1}f(y)d\,y + \lambda\frac{1}{\Gamma(\alpha)}\int_a^t (t-y)^{\alpha-1}g(y)d$$

$$\lambda D_a^{\alpha}[f(t) + g(t)] = \lambda D_a^{\alpha}f(t) + \lambda D_a^{\alpha}g(t) \quad (2.4)$$

2: $D_a^{o}f(t) = F(t)$ $I.e$ D_a^{o}=1 is the identity operator.

THEOREM 2:

$$D^{\alpha}[D_a^{-\beta}f(t)] = D_a^{-\beta}[D_a^{-\alpha}f(t)] = D_a^{-(\alpha+\beta)}f(t), \quad (2.5)$$

Where, $\alpha,\beta\epsilon R$ $(f(t) is\ continuous\ for\ t \geq 0$

PROOF: In this subsection we will derive rules for composition of fractional integrals. You can expect some problems due to the definition of the sequential derivative because differences between various sequences of Riemann-Liouville derivatives are its essence. First let us look at the composition of integrals because they are defined in the same way in both approaches. To point out the independence of the approach we will use the symbol D.We choose $a \in R, \alpha,\beta > 0, f(t)$ an integrable function. During the computation we use the change of order of integration and the Beta function.

Proof Let as using the definition of Mittag-Leffler function

$$E_{1,2}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty}\frac{x^k}{(k+1)!} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)!} = \frac{e^x-1}{x} \quad (1.12)$$

5: If $\alpha = 1$ and,$\beta = 3$ Than the Mittag-Leffler, function become

$$E_{1,3}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty}\frac{x^k}{(k+2)!} = \frac{1}{x^2}\sum_{k=0}^{\infty}\frac{x^{k+2}}{(k+2)!} = \frac{x^{k-1-x}}{x^2} \quad (1.13)$$

6: If $\alpha = 1$ and $\beta = 4$ than the Mittag-Leffler function become

$$E_{1,4}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+4)} = \sum_{k=0}^{\infty}\frac{x^k}{(k+3)!} = \frac{1}{x^3}\sum_{k=0}^{\infty}\frac{x^{k+3}}{(k+3)!} = \frac{e^x-x^2-2}{x^3} \quad (1.14)$$

In general

$$E_{1,m}(x) = \frac{1}{x^{m-1}}[e^x - \sum_{k=0}^{m-2}\frac{x^x}{k!} \quad (1.15)$$

Easily we can obtain the following result

$$E_{2,1}(x^2) = \sum_{k=0}^{\infty}\frac{x^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} = \cosh(x) \quad (1.16)$$

$$E_{2,2}(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k+1)!} = \frac{\sinh(x)}{x} \quad (1.17)$$

$$E_{2,1}(-x^2) = \sum_{k=0}^{\infty}\frac{(-x)^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} = \cos(x) \quad (1.18)$$

$$E_{2,2}(-x^2) = \sum_{k=0}^{\infty}\frac{(-x)^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{x(2k+1)!} = \frac{\sin(x)}{x} \quad (1.19)$$

FRACTIONAL INTEGRALS AND DERIVATIVES

2.1 FRACTIONAL INTEGRAL

There are more than one version of the fractional Integral and fractional derivative are exist. For example, it was touched above in the introduction that is the fractional integral be defined as follow as.

Let v be real non-integer number. Let f be piecewise continuous on $j' = (0,\infty)$ and integral any finite subinterval of $j = [0,\infty]$. There are $t > o$, we called (2.2) Riemann-Liouville Fractional Integral of order α .

In this section, we define the Cauchy's formula

$$D_a^t f(t) = \int_a^t\int_a^{t1}\int_a^{t2} \ldots\ldots \int_a^{tn-1} f(\tau)d\tau \ldots d\tau_2\, d\tau_1 = \frac{1}{\Gamma(n-1)}\int_a^t f(\tau)(t-\tau)^{n-1}d\,\tau \quad (2.1)$$

DEFINITION OF RIEMANN -LIOUVILLE FRACTIONAL INTEGRALS

Suppose that $\alpha > 0, t > \alpha, t, a\epsilon R\ then\ we\ have$

$$D_0^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x f(\tau)(x-\tau)^{\alpha-1}d\,\tau \quad (2.2)$$

The fractional Integration of a function to an arbitrary order α, and α is any non-negative real number. Then the equation (2.2) is called RIEMANN-LIOUVILLE Fractional Integral of order α

Other version of fractional integral is called WEYLE If

$$D_{-\infty}^t f(t) = \frac{1}{\Gamma f(\alpha)}\int_{-\infty}^t f(\tau)(t-\tau)^{\alpha-1}d\,\tau \quad (2.3)$$

2.2 SOME THEOREMS OF FRACTIONAL INTEGRATION

THEOREM: 1 $\lambda D_a^{\alpha}[f(t) + g(t)] = \lambda D_a^{\alpha}f(t) + \lambda D_a^{\alpha}g(t)$, $\alpha\epsilon R, \lambda\epsilon\mathbb{C}$,(Linear property)

$$D_a^{\alpha}x^u = x^{u+v}\frac{\Gamma(u+1)}{\Gamma(v+u+1)}$$

Hence we can calculate that

If $u = 0$ $than\ the\ above\ equation\ will\ come$ $x^u = x^0 = 1$

$than\ the\ equation$ (8)$the\ fractional\ integral\ of\ canstant$ "k" of order α is

$$D_a^{v}k = \frac{k!}{\Gamma(1+v)}x^v \quad ,v > 0$$

$In\ particular\ form\ If$ $v = \frac{1}{2}$ Than $D^{\frac{-1}{2}}x^0 = \frac{1}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}$

2.3: FRACTIONAL DERIVATIVES

We are introduce the notation of the Fractional Integrals $D_0^{\alpha}f(x)$ we denoted the Fractional Derivatives of a function $f(x)$ to an arbitrary order $v > 0$. The Fractional Derivatives can be defined using the definitions of the Fractional Integrals to this end, suppose that$v = n - \alpha$, Where, $0 < v < 1$and n this smallest integer greater than α.Than, the Fractional D of$f(x)$ of order α

$$D_0^{\alpha}f(x) = D^n[D^{-v}f(x)]$$

The fractional derivative can be defined in terms of fractional integral fractional derivative into divided two parts

1: Suppose that $\alpha > 0,\ t > a,, a, t, \epsilon R.\ than\ we\ have$

$$D^{\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(n-1)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau\ , & n-1 < \alpha < n\epsilon N \\ \frac{d^n}{dt^n}f(t), & \alpha = n\epsilon N \end{cases} \quad (2.8)$$

$$D_a^{-\alpha}\left(D_a^{-\beta}f(t)\right)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\xi)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_a^{\xi}(\xi-\tau)^{\beta-1}f(\tau)d\tau\right)d\xi$$

$$D_a^{-\alpha}\left(D_a^{-\beta}f(t)\right)=\frac{1}{\Gamma(\alpha)\Gamma\beta}\int_a^t(t-\xi)^{\alpha-1}(\xi-\tau)^{\beta-1}f(\tau)d\tau d\xi$$

Change of order of integration

Let $\frac{\xi-\tau}{t-\tau}=z$, $d\tau=(t-\tau)dz$

$$D_a^{-\alpha}\left(D_a^{-\beta}f(t)\right)=\frac{1}{\Gamma(\alpha)\Gamma\beta}\int_a^t f(\tau)\int_\tau^t(t-\xi)^{\alpha-1}(\xi-\tau)^{\beta-1}\,d\xi d\tau$$

$$D_a^{-\alpha}\left(D_a^{-\beta}f(t)\right)=\frac{1}{\Gamma(\alpha)\Gamma\beta}\int_a^t f(\tau)(t-\tau)^{\alpha+\beta-1}\beta(\alpha,\beta)d\tau$$

$$D_a^{-\alpha}\left(D_a^{-\beta}f(t)\right)=\frac{1}{\Gamma(\alpha+\beta)}\int_a^t f(\tau)(t-\tau)^{\alpha+\beta-1}\beta(\alpha,\beta)d\tau=D_a^{\alpha+\beta}f(t)\ ,$$

So we, just proved that fractional integrals are commutative (exactly the same result we, have

$$D_a^{-\alpha}\left(D_a^{-\beta}f(t)\right)=D_a^{\alpha+\beta}f(t)\ \ ,a\epsilon R,\ \ \alpha,\beta>0 \qquad (2.6)$$

One of the most important theorem in development of fractional calculus is the law of exponents in

In which is very useful to calculate fractional integral and fractional derivative.

THEOREM 3: let F be continuous function on J *and* *let* $u,\upsilon>0$ *than for all to than , we , have*

$$D^{-u}[D^{-\upsilon}\text{f(t)}]=D^{-(u+\upsilon)}f(t)=D^{-\upsilon}[D^{-u}f(t)]$$

Another useful property in the study of fractional calculus is the commutative property.

THEOREM 4:let F be continuous on j and

$$D[D^{-\upsilon}f(t)]=D^{-\upsilon}\,[Df(t)]+\frac{f(t)|t=0}{\Gamma(\upsilon)}t^{\upsilon-1} \qquad (2.7)$$

Thus in general

$$D[D^{-\upsilon}f(t)]\neq D^{-\upsilon}\,[Df(t)]$$

EXAMPLE: Let evaluate $D_a^{\alpha}x^u$ where, $u>1$ *and* $\alpha>0$

Solution: According to the definition of Riemann-Liouville fractional integral

$$D_0^t\text{f(t)}=\frac{1}{\Gamma(\alpha)}\int_a^t f(\tau)(t-\tau)^{\alpha-1}d\,\tau$$

$$D_a^{\alpha}x^u=\frac{1}{\Gamma f(\alpha)}\int_a^x(x-t)^{\alpha-1}t^u\,dt$$

$$D_a^{\alpha}x^u=\frac{1}{\Gamma(\alpha)}\int_0^x x^{\alpha-1}\,(1-\frac{t}{x})^{\alpha-1}t^u\,dt$$

Substitute $\frac{t}{x}=\text{u}$ then xu = t , udx = dt

$$D_a^{\alpha}x^u\ =\frac{1}{\Gamma(\alpha)}\int_0^1 x^{\alpha-1}(1-u)^{\alpha-1}(ux)^u u dx$$

$$D_a^{\upsilon}x^u=\ \frac{1}{\Gamma(\upsilon)}x^{u+\alpha}\int_0^1(1-u)^{\upsilon-1}u^u du$$

Using the definition of beta function

$$D_a^{\upsilon}x^u\ =\frac{1}{\Gamma(\upsilon)}x^{u+\alpha}\beta(\upsilon,u+1)$$

Again using the relation between Beta and Gamma functions

$$D_a^{\upsilon}x^u\ =\frac{1}{\Gamma(\upsilon)}x^{u+\alpha}\frac{\Gamma(u+1)\Gamma(\upsilon)}{\Gamma(\alpha+u+1)}$$

$$D_a^t\text{f(t)}=\frac{1}{\Gamma(n-1)}\int_a^t f(\tau)(t-\tau)^{n-1}d\,\tau \qquad (2.11)$$

Then we have

$$R_{n-1}=\int_0^t\frac{f^{(n)}(\tau)(t-\tau)^{n-1}}{(n-1)!}d\tau=\frac{1}{\Gamma(n)}\int_0^t f^{(n)}(\tau)(t-\tau)^{n-1}d\tau=j^n f^{(n)}(t)$$

(2.12)

Now n, by using the linearity characteristic of the Riemann-Liouville fractional derivative, we obtain

$$D^{\alpha}f(t)=D^{\alpha}\left(\sum_{k=0}^{n-1}\frac{t^k}{\Gamma(k+1)}f^{(k)}(0)+R_{n-1}\right) \qquad (2.13)$$

$$D^{\alpha}f(t)=\sum_{k=0}^{n-1}\frac{D^{\alpha}t^k}{\Gamma(k+1)}f^{(k)}(0)+D^{\alpha}R_{n-1}$$

$$D^{\alpha}f(t)\ =\sum_{k=0}^{n-1}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\frac{t^{k-\alpha}}{\Gamma(k+1)}f^{(k)}(0)+D^{\alpha}j^n f^n(t)$$

$$D^{\alpha}f(t)=\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0)+j^{n-1}f^n(t)$$

$$D^{\alpha}f(t)=\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0)+D_*^{\alpha}f(t)$$

Thus mean that

$$D_*^{\alpha}f(t)=D^{\alpha}f(t)-\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0)$$

The proof is complete.

THEOREM 2. The Riemann-Liouville fractional derivative of the power function satisfies.

$$D^{\alpha}t^r=\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}t^{r-\alpha}\quad n-1<\alpha<n,\ r>-1,\ r\epsilon R \qquad (2.14)$$

This is named the Riemann-Liouville fractional derivative of order α.

2:

Suppose that$\alpha>0,\ t>a,,a,t,\epsilon R$. The fractional Captuo operator has the form

$$D_*^{\alpha}f(x)=\begin{cases}\frac{1}{\Gamma(n-1)}\frac{d^n}{dt^n}\int_a^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau\ , & n-1<\alpha<n\epsilon N\\ \frac{d^n}{dt^n}f(t), & \alpha=n\epsilon N\end{cases} \qquad (2.9)$$

Remark: The difference between Caputo and Riemann – Liouville formulas for the fractional derivatives leads to the following differences

A: Caputo fractional derivative of a constant equals zero, while Riemann – Liouville fractional derivative of a constant does not equal zero

B:The non-commutation, in Caputo fractional derivative we have

$$D_a^{\alpha}D^m f(t)=D_a^{\alpha+m}f(t)\neq D_a^{\alpha}f(t)$$

Where $\alpha\epsilon(n-1,n),n\epsilon N,m=1,2\,...$

In general, the Riemann-Liouville derivative is also non-commutation as

$$D_a^{\alpha}D^m f(t)=D_a^{\alpha+m}f(t)\neq D_a^{\alpha}D_a^m f(t)$$

Where $\alpha\epsilon(n-1,n),n\epsilon N,m=1,2,...$

2.4 FRACTIONAL DERIVATIVE OF THE POWER FUNCTION

In order to comprehension the fractional derivative of the power function, we review

Some theorems related to our work.

THEOREM 1

Suppose $t>0,\alpha\epsilon R$ and $n-1<\alpha<n\ ,n\epsilon N,$ than the following Riemann-Liouville and Caputo operator hold.

$$D_*^{\alpha}f(t)=D^{\alpha}f(t)=\sum_{k=0}^{n=1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0) \qquad (2.10)$$

PROOF: The well-known The Taylor series exponential about the point t=0

$$f(t)=f(0)+tf'(0)+\frac{t^2}{2!}f''(0)+\frac{t^3}{3!}f'''(0)+\cdots+\frac{t^{n-1}}{(n-1)!}f^{n-1}(0)+R_{n-1.}$$

$$=\sum_{k=0}^{n-1}\frac{t^k}{\Gamma(k+1)}f^k(0)+R_{n-1}$$

Considering

$$D_a^t\text{f(t)}=\int_a^t\int_a^{t_1}\int_a^{t2}\,......\int_a^{tn-1}f(\tau)d\tau\,...\,d\tau_2\,d\tau_1$$

$$D_*^{\alpha}t^p\ =\frac{\Gamma(p+1)}{\Gamma(p-n+1)}t^{p-\alpha}$$

PROOF of the first case

The second case

$$D_*^{\alpha}t^p=0 \qquad (2.17)$$

When $n-1<\alpha<n,p\leq n-1,p\epsilon\{-1,-2,-3,...\})$ The following the design of the proof of the differential of the constant, since $(t^{p^n})=0$

For $p\leq n-1,p,n\epsilon N$.so.The proof of the theorem is complete.

THEOREM 4. Let $f(t)=t^u$ where $u>1$, $t>0$ and Re $\upsilon>0$ Then the Riemann- Liouville Fractional Integral power function, satisfy

$$D^{-\upsilon}f(t)=\frac{\Gamma(u+1)}{\Gamma(\upsilon+u+1)}t^{u+\upsilon} \qquad (2.18)$$

PROOF 4: use the definition of the Riemann-Liouville of fractional integral of the power function satisfies

$$D^{-\upsilon}f(t)=\frac{1}{\Gamma(\upsilon)}\int_0^t f(\xi)(t-\xi)^{\upsilon-1}d\,\xi$$

$$D^{-\upsilon}f(t)=\frac{1}{\Gamma(\upsilon)}\int_0^t(t-\xi)^{\upsilon-1}\xi^u d\,\xi$$

$$D^{-\upsilon}f(t)=\frac{t^{\upsilon-1}}{\Gamma(\upsilon)}\int_0^t\ (1-\frac{\xi}{t})^{\upsilon-1}\xi^u\,d\xi$$

$$D^{-\upsilon}f(t)\ \ =\frac{t^{\upsilon}}{\Gamma(\upsilon)}\int_0^t\ (1-\frac{\xi}{t})^{\upsilon-1}\xi^u\frac{1}{t}d\xi$$

By some substitution $y=\frac{\xi}{t}$

$$D^{-\upsilon}f(t)=\frac{t^{\upsilon+u}}{\Gamma(\upsilon)}\int_0^t\ (1-y)^{\upsilon-1}(\frac{\xi}{t})^{\mu}\,dy$$

$$D^{-\upsilon}f(t)\ =\frac{t^{\upsilon+u}}{\Gamma(\upsilon)}\int_0^t\ (1-y)^{\upsilon-1}y^{\mu}\,dy$$

$$D^{-\upsilon}f(t)\ =\frac{\Gamma(\upsilon)\Gamma(\mu+1)t^{\upsilon+u}}{\Gamma(\upsilon+\mu+1)\Gamma(\upsilon)}\int_0^t\frac{\Gamma(\upsilon+\mu+1)}{\Gamma(\upsilon)\Gamma(\mu+1)}\,(1-y)^{\upsilon-1}y^{\mu}\,dy$$

$$\text{D}^{-\upsilon}\text{f(t)}=\frac{\Gamma(u+1)}{\Gamma(\upsilon+u+1)}t^{u+\upsilon}$$

So, the proof of the theorem is complete.

PROOF 2: we have to use the definition of Riemann-Liouville fractional integral

$$D^{\alpha}t^{r} = D^{n}\left[D^{-(n-\alpha)}t^{\alpha}\right]$$

$$D^{\alpha}t^{r} = \frac{1}{\Gamma(\alpha-1)}\int_0^t t^{r}(t-x)^{n-\alpha-1}dx$$

$$D^{\alpha}t^{r} = \frac{1}{\Gamma(\alpha-1)}\int_0^x (t-x)^{n-\alpha-1}t^{r}\,dx$$

$$D^{\alpha}t^{r} = \frac{1}{\Gamma(\alpha-1)}\int_0^x x^{\alpha-1}\,(1-\tfrac{t}{x})^{n-\alpha-1}t^{r}\,dt$$

Substitute $\frac{t}{x} = \mathrm{u}$ then $\mathrm{xu} = \mathrm{t}$, $\mathrm{udx} = \mathrm{dt}$

$$D^{\alpha}t^{r} = \frac{1}{\Gamma(\alpha-1)}\int_0^1 x^{n-\alpha-1}(1-u)^{n-\alpha-1}(ux)^{r}udx$$

$$D^{\alpha}t^{r} = \frac{1}{\Gamma(\alpha-1)}x^{r-\alpha}\int_0^1(1-u)^{\upsilon-1}u^{u}du$$

Using the definition of beta function

$$D^{\alpha}t^{r} = \frac{1}{\Gamma(\alpha-1)}x^{r-\alpha}\beta(\upsilon,u+1)$$

Again using the relation between Beta and Gamma functions

$$D^{\alpha}t^{r} = \frac{1}{\Gamma(\alpha-1)}x^{r-\alpha}\frac{\Gamma(r+1)\Gamma(\alpha-1)}{\Gamma(r-\alpha+1)}$$

$$D^{\alpha}t^{r} = x^{r-\alpha}\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} \qquad (2.15)$$

Where $n-1<\alpha<n,\ r>-1,r\epsilon R$

THEOREM 3. The Caputo fractional derivative of the power function is

$$D_*^{\alpha}t^{p} = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha} & n-1<\alpha<n, p>n-1, p\epsilon R \\ 0 & n-1<\alpha<n, p>n-1, p\epsilon\{-1,-2,-3\ldots.\} \end{cases} \qquad (2.16)$$

PROOF 3: For $n-1<\alpha<n, p>n-1, p\epsilon R$,

$$D_*^{\alpha}t^{p} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{(\tau^{p^{n}})}{(t-\tau)^{\alpha-1-n}}d\tau$$

$$D_*^{\alpha}t^{p} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\Gamma(p+1)}{\Gamma p-n+1)}(\tau^{p-n})(t-\tau)^{n-\alpha-1}d\tau$$

And using substitution $\tau = \lambda t$ $,0\leq\lambda\leq 1$

$$D_*^{\alpha}t^{p} = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}\int_0^t(\lambda t)^{p-n}((1-\lambda)t)^{n-\alpha-1}td\lambda$$

$$D_*^{\alpha}t^{p} = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}t^{p-\alpha}\int_0^t\lambda^{p-n}(1-\lambda)^{n-\alpha-1}td\lambda$$

$$D_*^{\alpha}t^{p} = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}t^{p-\alpha}\beta(\mathrm{p-n+1,n-\alpha})$$

$$D_*^{\alpha}t^{p} = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}t^{p-\alpha}\frac{\Gamma(p-n+1)\Gamma(n-\alpha)}{\Gamma(p-\alpha+1)}$$

$$\mathrm{D}^{1/2}\mathrm{t}^{6} = \frac{\Gamma(6+1)}{\Gamma(6-\frac{1}{2}+1)}\mathrm{t}^{6-\frac{1}{2}} = \frac{6!}{\Gamma(\frac{13}{2})}\mathrm{t}^{\frac{11}{2}} = \frac{46080}{10395\sqrt{\pi}}\mathrm{t}^{\frac{11}{2}}$$

The fourth case: when $p=7$ and $\alpha=\frac{1}{2}$

$$D^{1/2}t^{7} = \frac{\Gamma(7+1)}{\Gamma(7-\frac{1}{2}+1)}t^{7-\frac{1}{2}} = \frac{7}{\Gamma(\frac{15}{2})}\mathrm{t}^{\frac{13}{2}} = \frac{465120}{135135\sqrt{\pi}}\mathrm{t}^{\frac{13}{2}}$$

EXAMPLE: 2.3

In this example we will apply the form $D^{-\alpha}t^{u} = \frac{\Gamma(u+1)}{\Gamma(\mu+\upsilon+1)}\mathrm{t}^{\mathrm{u+\upsilon}}$, on the result previous four cases, in example (2) respectively then we getting $f(t)$ as follows

Fist case

$$\mathrm{D}^{-3/2}\frac{192}{15\sqrt{\pi}}\mathrm{t}^{\frac{5}{2}} = \frac{192}{15\sqrt{\pi}}\left(\frac{\Gamma(\frac{5}{2}+1)}{\Gamma(\frac{5}{2}+\frac{3}{2}+1)}\right)\mathrm{t}^{\frac{5}{2}+\frac{3}{2}}$$

$$= \frac{192}{15\sqrt{\pi}}\left(\frac{\Gamma(\frac{7}{2})}{\Gamma(5)}\right)\mathrm{t}^{4} = \frac{192}{15\sqrt{\pi}}\left(\frac{15\sqrt{\pi}}{192}\right)\mathrm{t}^{4} = \mathrm{t}^{4} = \mathrm{f(t)}$$

APPLICATIONS

In this section we presen and fractional integral, in order to comprehension the Riemann the power function.

EXAMPLE: 2.1

Suppose that $f(t) = t^{u} = k$, and k is constant when we apply the form

$$D^{-\alpha}f(t) = \frac{\Gamma(\mathrm{u}+1)}{\Gamma(\mu+\alpha+1)}\mathrm{t}^{\mathrm{u}+\alpha}, \alpha, \mathrm{t}, > 0, \mu > -1$$

we will obtian .

$$D^{-\alpha}f(t) = \frac{k}{\Gamma(\alpha+1)}\mathrm{t}^{\alpha}$$

EXAMPLE: 2.2

Suppose that $f(t) = t^{\mu}$ when $n-1<\alpha<n, p>n-1, p\epsilon R$ $t>$ (α *is* order differentiation).Four cases are consider, namely f derivative of the functions t^4, t^5, t^6 and t^7 I.e., $p=4$, $p=5$, $p=6$,

$p=7$, respectively, by applying the form

$$D^{\alpha}f(t) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\mathrm{t}^{\mathrm{p}-\alpha}$$

We, will obtain

The first case If $p=4$, *and* $\alpha=\frac{3}{2}$

$$D^{3/2}t^{4} = \frac{\Gamma(4+1)}{\Gamma(4-\frac{3}{2}+1)}\mathrm{t}^{4-\frac{3}{2}} = \frac{4!}{\Gamma(\frac{7}{2})}\mathrm{t}^{\frac{5}{2}} = \frac{192}{15\sqrt{\pi}}\mathrm{t}^{\frac{5}{2}}$$

The second case. When $p=5$ and $\alpha=\frac{3}{2}$

$$\mathrm{D}^{3/2}\mathrm{t}^{5} = \frac{\Gamma(5+1)}{\Gamma(5-\frac{3}{2}+1)}\mathrm{t}^{5-\frac{3}{2}} = \frac{5!}{\Gamma(\frac{9}{2})}\mathrm{t}^{\frac{7}{2}} = \frac{1920}{105\sqrt{\pi}}\mathrm{t}^{\frac{5}{2}}$$

The third case: If $p=6$, *and* $\alpha=\frac{1}{2}$ Than

Second case

$$\mathrm{D}^{-3/2}\frac{1920}{105\sqrt{\pi}}\mathrm{t}^{\frac{5}{2}} = \frac{1920}{105\sqrt{\pi}}\left(\frac{\Gamma(\frac{7}{2}+1)}{\Gamma(\frac{7}{2}+\frac{3}{2}+1)}\right)\mathrm{t}^{\frac{7}{2}+\frac{3}{2}}$$

$$\frac{1920}{105\sqrt{\pi}}\left(\frac{\Gamma(\frac{9}{2})}{\Gamma(6)}\right)\boldsymbol{t}^{5} = \frac{1920}{105\sqrt{\pi}}\left(\frac{105\sqrt{\pi}}{1920}\right)\mathrm{t}^{5} = \mathrm{t}^{5} = \mathrm{f(t)}$$

The third case

$$\mathrm{D}^{-1/2}\frac{46080}{10395\sqrt{\pi}}\mathrm{t}^{\frac{11}{2}} = \frac{46080}{10395\sqrt{\pi}}\left(\frac{\Gamma(\frac{11}{2}+1)}{\Gamma(\frac{11}{2}+\frac{3}{2}+1)}\right)t^{\frac{11}{2}+\frac{1}{2}}$$

$$\frac{46080}{10395\sqrt{\pi}}\left(\frac{\Gamma(\frac{13}{2})}{\Gamma(7)}\right)t^{6} = \frac{46080}{10395\sqrt{\pi}}\left(\frac{10395\sqrt{\pi}}{46080}\right)t^{6} = t^{6} = f(t)$$

The fourth case

$$\mathrm{D}^{-1/2}\frac{465120}{135135\sqrt{\pi}}\mathrm{t}^{\frac{13}{2}} = \frac{465120}{135135\sqrt{\pi}}\left(\frac{\Gamma(\frac{13}{2}+1)}{\Gamma(\frac{13}{2}+\frac{1}{2}+1)}\right)\boldsymbol{t}^{\frac{13}{2}+\frac{1}{2}}$$

$$= \frac{465120}{135135\sqrt{\pi}}\left(\frac{135135\sqrt{\pi}}{465120}\right)\mathrm{t}^{7} = \mathrm{t}^{7} = \mathrm{f\,(t)}$$

CONCLUSIONS

Fractional Calculus was formulated in 1695, shortly after the development of classical calculus. The earliest systematic studies were attributed to Liouville, Riemann, Leibniz, etc. For a long time, fractional calculus has been regarded as a pure mathematical realm without real applications. But, in recent decades, such a state of affairs has been changed. It has been found that fractional calculus can be useful and even powerful, and an outline of the simple history about fractional calculus, especially with applications, can be found in Machado et al. .

Now, fractional calculus and its applications is undergoing rapid developments with more and more convincing applications in the real world.

The use of fractional order derivatives is nowadays widespread in many fields. Indeed, the Possibility to use any real order improves the modelling of several phenomena in physics, engineering and many application areas.

The subject of fractional differential equations is gaining much importance and attention. The so-called fractional differential equations are specified by generalizing the standard integer order derivative to arbitrary order. Fractional differential equations (FDEs) involve fractional derivatives of the form (d^{α}/dx^{α}), which are defined for $\alpha>0$, where α is not necessarily an integer. They are generalizations of the ordinary differential equations to a random (non-integer) order. They have attracted considerable interest due to their ability to model complex phenomena. Due to the effective memory function of fractional derivative, fractional differential equations have been widely used to describe many physical phenomena such as flow in porous media and in fluid dynamic traffic model. For more interesting theory results and scientific applications of fractional differential equations, we cite the monographs of Diethelm , Kilbas et al. , Hilfer , Miller and Ross , Podlubny , Zhou and the references therein.

REFERENCES

[1] Miller, K.S.; Ross, B. An Introduction to the Fractional Calculus and Fractional Differential Equations; John Wiley and Sons: New York, NY, USA, 1993.

[2] Baleanu D., Muslish S. I., Tas K., "Fractional Hamiltonian Analysis of Higher Order Derivatives Systems", Journal of Mathematical Physics, 147(10), 103503, (2006).

[3] Podlubny Igor. Fractional Differential Equations. United States Academy Press C1999.30 p. ISBN 0-12-558840-2.

[4] Kilbas Anatoly A,Srivastava, Hari M,Trojilo ,Juan J, Theory and Applications of Fractional Differential Equations . John van Mill. Nethelands , Elsevier, 2006,523 p, ISBN 978—0444-51832-3.

[5] Caputo M., Fabrizio M., "A New Definition of fractional Derivative without Singular Kernel", Progress in Fractional Differentiation and Applications, 1(2), 74 (2015).

[6] Podlubny I., "Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation ", Fractional Calculus and Applied Analysis, 5(4), 367 (2002).

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[7] Corl F.Lorenzo & Tom T.Hartly, Generalized Functions for the Fractional Calculus, Nasa/TP-1999-209424/REV1, National Aeronautics and Space Administration, Glenn Research Center, October, 1999.

[8] Damian P. Watson, Fractional Calculus and its Applications, Department of Mathematics , Mary Washington College Fredericksburg, VA 22401, April 27, 2004.